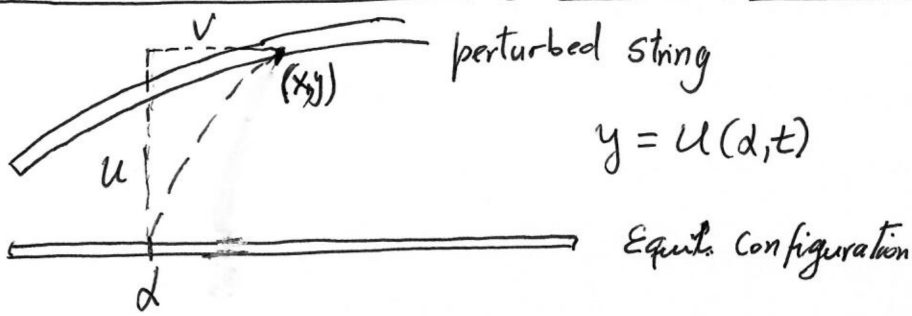
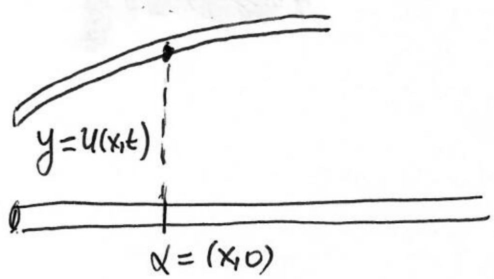
4.2 Derivation of a Vertically Vibrating String.



Assumption: $|v| << 1$ Very small. It can be neglected.

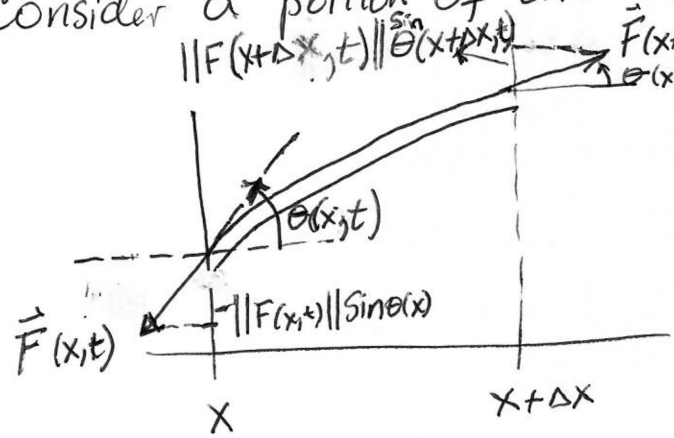
Since slope of perturbed string is small

The previous assumption leads to entire vertical motion.



Application of Newton 2nd Law:

Consider a portion of the string



Internal forces or Tensions in the string.

$[Q(x,t)] = \frac{F}{M}$

If $\|\vec{F}(x,t)\| = T(x,t)$, $Q(x,t)$: Vertical Component of body forces/mass

Then $m\vec{a} = \sum \vec{F}$

$\rho_0(x)$: Linear density (mass) $[\rho_0] = \frac{M}{L}$

implies

[Newton 2nd Law applied to portion of string $[x, x+\Delta x]$]

$$\int_x^{x+\Delta x} \rho_0(x) \frac{\partial^2 u}{\partial t^2}(x,t)\,dx = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + \int_x^{x+\Delta x} \rho_0(x)\,Q(x,t)\,dx$$

Using the Mean Value theorem for integrals: $\bar{x} \in (x, x+\Delta x)$

$$\rho_0(\bar{x}) \frac{\partial^2 u}{\partial t^2}(\bar{x},t)\Delta x = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x) + \rho_0(\bar{x})\,Q(\bar{x},t)\Delta x$$

Dividing by $\rho_0(\bar{x})\Delta x$ and taking $\lim_{\Delta x \to 0}$.

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(T\sin\theta\right)(x,t) + \rho_0(x)\,Q(x,t)$$

Now, for small angles θ.

$$\frac{\partial u}{\partial x} = \tan\theta \simeq \frac{\sin\theta}{\cos\theta} \simeq \sin\theta$$

Then, we obtain

$$\boxed{\rho_0(x)\,\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right)(x,t) + \rho_0(x)\,Q(x,t)} \quad (3.1)$$

<u>Assumption</u>: Perfectly elastic string and θ small

then, $T(x,t) \simeq T_0$ constant.

and

$$\boxed{\rho_0(x)\,u_{tt}(x,t) = T_0\,u_{xx}(x,t) + \rho_0(x)\,Q(x,t)} \quad (3.2)$$

If $Q(x,t) = -g$ and $\rho_0 g << |T_0 u_{xx}|$

then this body force can be neglected.

and we arrive to

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0(x)}\frac{\partial^2 u}{\partial x^2} = c^2(x)\,\frac{\partial^2 u}{\partial x^2}} \quad (3.3)$$

One-dimensional wave equation.

Show that $[c(x)] = \frac{L}{t}$.

If $Q(x,t)=0$, then

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2_{(x)} \frac{\partial^2 u}{\partial x^2}}, \quad c^2_{(x)} \equiv T_0/\rho_0(x).$$

$\rightarrow$ wave speed

$$\left[\frac{T_0}{\rho_0}\right] = \frac{M\frac{L}{t^2}}{\frac{M}{L}} = \frac{L^2}{t^2} \checkmark$$

If string is uniform then c is constant.

One-dimensional wave equation.

4.3

Boundary conditions:

a) Fixed ends: $\begin{cases} u(0,t)=0 \\ u(L,t)=0. \end{cases}$

b) Prescribed Variations: $\begin{cases} u(0,t)=f(t) \\ u(L,t)=g(t) \end{cases}$

c) Free ends: $u_x(0,t)=0$, $u_x(L,t)=0$

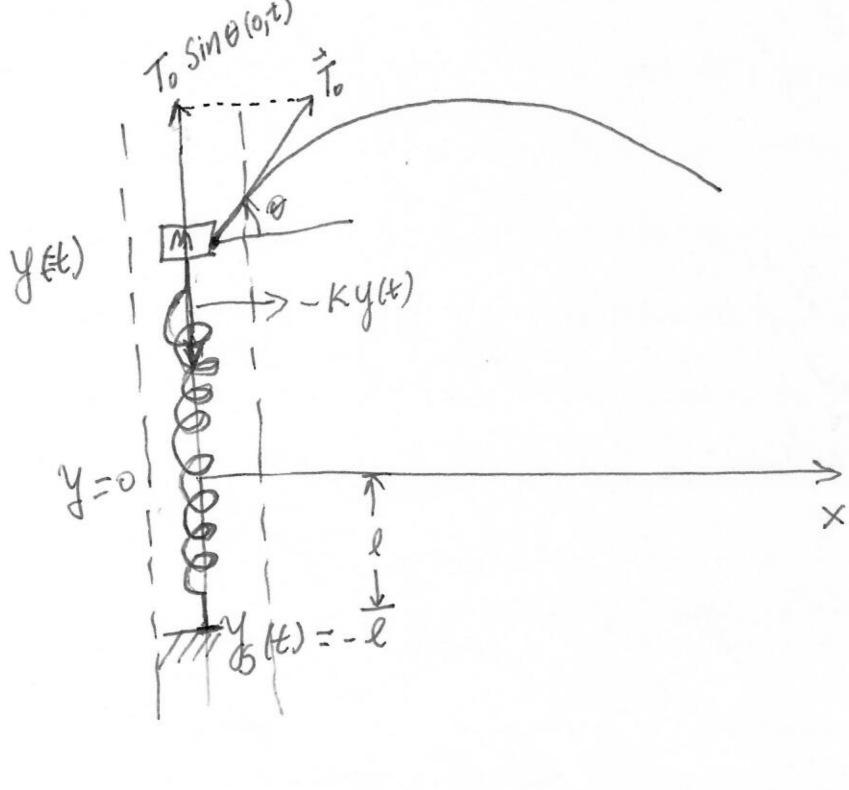
Initial conditions:

$u(x,0)=\phi(x)$

$u_t(x,0)=\psi(x)$

Why two IC's?

c) Left end $x=0$ is attached to a Spring-mass system.



ℓ: Spring natural length.

$y(t) = u(0,t)$. B.C.